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Analysis of Students' Difficulties in Solving Integration Problems[1]

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Abstract: Secondary school students often experience difficulty with solving integration problems. In this study, designed to examine such difficulties, data collection involved the use of a six-question test followed by interviews of selected students. With respect to students' performance, they had difficulty with questions that involved integration of trigonometric functions and applying integration to evaluate plane areas. The students seemed to focus more on the procedural aspects of integration than on the conceptual aspects. They generally lacked both conceptual and procedural understanding of integration. The largest numbers of errors committed were technical errors which were primarily attributed to the students' lack of specific mathematical content knowledge.

Introduction

Integration is part of the Additional Mathematics syllabus required for the Singapore-Cambridge General Certificate in Education Ordinary Level Examinations. Even after the content reduction by the Singapore Ministry of Education, integration is still relevant in the revised syllabus for year 2001 and beyond (University of Cambridge Local Examinations Syndicate, 2001). For 15 to 16 year olds in the upper secondary grades, this topic is usually covered in their Secondary Four year right after they finished the topic on differentiation. However, although integration is an important topic in the Additional Mathematics syllabus, students generally find it difficult to cope with and encounter various difficulties while solving integration problems (Seah, 2003). It is the purpose of this study to investigate the nature of these difficulties.

In Singapore, research into the area of calculus teaching and learning has been ongoing. For example, in 1997, a research project called "Calculus Education at the Junior College and Tertiary Levels in Singapore" was initiated and funded by the National Institute of Education of the Nanyang Technological University (Ahuja, Lee, Lim-Teo, Tan, & Chua, 1998). Principal findings from the project included that, although students generally had a positive attitude towards calculus, they just

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wanted to memorise formulae to solve problems with minimal understanding as they did not see understanding of concepts as important for getting high grades in mathematics and calculus; that some university students encountered difficulties in learning calculus, especially with regards to concepts, definitions, theorems and proofs; and, students generally found calculus courses boring.

Studies identifying mathematics difficulties experienced while students solve mathematical problems are not new. For example, Donaldson (1963) described three types of errors that occurred while students learn mathematics. The errors included (a) *structural* due to failure to appreciate the relationships involved in the problem; (b) *arbitrary* errors arising from the student failing to take account of the constraints laid down in what was given; and (c) *executive* involving failure to carry out manipulations despite having understood the principles involved. Avital and Libeskind (1978) also described three types of difficulties students encountered in mathematical induction; namely conceptual, mathematical and technical difficulties. Subsequently, Chow (2002) did a study on mathematical induction based on Avital and Libeskind's classification of difficulties on 30 junior college students in Singapore and made similar conclusions to those of Avital and Libeskind.

There are studies that focus specifically on students' difficulties with learning of calculus. For example, Orton (1983a) did a study using Donaldson's (1963) system of classification of errors on 60 high school and 50 college students. He devised a clinical interviewing method to investigate students' understanding of elementary calculus. Students' responses to tasks concerning integration and limits were analysed in detail. From the data obtained, students' degree of understanding and the common errors and misconceptions were found. Generally, students had problem with the understanding of integration as the limit of a sum, and the relationship between a definite integral and areas under the curve. According to him, many teachers have accepted the fact that integration cannot be made easy and have responded in a variety of ways. Some teachers reacted by introducing integration as a rule or as anti-differentiation while others tried to build up understanding of limits and background algebra before introducing integration. More recently, Thomas and Ye (1996) did a study associated with integration on students' processes and concepts. The purpose of their study was to investigate student thinking and misconceptions when dealing with the Riemann integral. They found that students lacked certain conceptual understanding and were often engaged in an instrumental, process-oriented style of thinking which hindered their understanding of important concepts.

A number of these calculus-based studies are however dated and have been done outside of Singapore. Thus, research based on the syllabus required for the

Singapore-Cambridge General Certificate in Education Ordinary Level Examinations to identify the types of questions on integration that Singapore students can do, the kind of questions that students face difficulty dealing with, and the reasons behind students' weaknesses is not extensive. It is the purpose of this study to collect and analyse such students' difficulties in solving integration problems. The following research questions were considered:

1. Given the various objectives with respect to integration in the calculus component of the Additional Mathematics syllabus, what is the degree of understanding of integration concepts among Secondary Four school students in Singapore?

2. What are common errors and misconceptions that secondary school students have with respect to integration?

To accomplish this purpose, and based on my analysis of the work of Donaldson (1963), Avital and Libeskind (1978), and Orton (1983a), I developed a conceptual framework shown in Figure 1 to classify the different possible errors and misconceptions that students may encounter while solving integration problems. Possible errors that may be made by students were classified into three categories. The first type of error is *conceptual* error. It refers to errors due to failure to grasp the concepts involved in the problem or errors that arise from failure to appreciate the relationships involved in the problem. The second type of error is *procedural*. Procedural errors are those which arise from failure to carry out manipulations or algorithms despite having understood the concepts behind the problem. The third type of error, *technical* error, refers to errors due to a lack of mathematical content knowledge in other topics or errors due to carelessness. It was anticipated that some students may make both conceptual and procedural errors in a single problem, or have misconceptions about a problem but still manage to get the correct answer. Hence, in the analysis of students' errors, it had to be recognised that this conceptual framework is subject to adjustment based on actual data obtained.

Research Methodology

Subjects

This study was conducted in the first half of 2003 in a secondary school located in the western part of Singapore with a total student population of around 1000. For the data collection, a test was designed and administered. Forty Secondary Four students (16 years of age) did the test. The 40 students consisted of 15 males and 25 females. These students were from the only class in the whole level that was taking both Mathematics Syllabus D and Additional Mathematics for the GCE 'O' Level Examinations. The subjects had a mean score of 206 out of a maximum of 300 in

Types of Errors	Description
Conceptual Error	• Failure to grasp the concepts in problem. • Errors from failure to appreciate the relationships in problem. Example: Area between the curve y = x(x – 4) and the x-axis from x = 0 to x = 5 is: $\int_0^5 x(x-4)dx = \int_0^5 (x^2-4x)dx$ $= -8\frac{1}{3}\ units^2$ Students fail to realize that the part of the curve y = x(x – 4) from x = 0 to x = 4 is below the x-axis whereas the part from x = 4 to x = 5 is above the x-axis.
Procedural Error	• Errors from failure to carry out manipulations or algorithms although concepts in problem are understood. Example: $\int \tan^2 2x dx = \int (\sec^2 2x - 1)dx$ = tan 2x – x + c Students fail to put a coefficient of $\frac{1}{2}$ in front of tan 2x.
Technical Error	• Errors due to lack of mathematical content knowledge in other topics. • Errors due to carelessness. Example: $\int 2(3+4x)^4 dx = \int (6+8x)^4 dx$ $= \left[\frac{(6+8x)^5}{5\times 8}\right] + c$ $= \frac{(6+8x)^5}{40} + c$ Students wrongly multiplied the constant of 2 into the binomial before integrating.

Figure 1. Classification of students' errors

their Primary School Leaving (PSLE) Examination. As for their mathematical abilities, most of the students scored either an A or a B for their mathematics in their PSLE Examination. Thus within the school context these students may be considered mathematically the best, whereas in a national sense, they may be regarded as of medium or mixed ability.

Instrumentation

The test used for the study consisted of six questions which were carefully selected from various sources:

(a) past-year questions from General Certificate in Education Ordinary Level Examinations,
(b) past-year Preliminary Examination questions from various secondary schools, and
(c) questions from textbooks used in other countries.

The six questions tested students on the four objectives in the syllabus pertaining to integration (University of Cambridge Local Examinations Syndicate, 2001):

- integrating sums of terms in powers of x excluding $\frac{1}{x}$;
- integrating polynomial, trigonometric, and exponential functions;
- evaluating definite integrals and apply integration to the evaluation of plane area; and
- applying differentiation and integration to kinematics problems that involve displacement, velocity and acceleration of a particle moving in a straight line with variable or constant acceleration, and the use of *x*-*t* and *v*-*t* graph.

The test questions are presented in the Appendix.

Data collection and Analysis

Data collection involved two stages: a test and interviews. The test was proofread by the Head of Mathematics Department of the school before being administered under normal testing conditions over two 35-minute periods. The subjects were told in advance of the test and had ample time to prepare for it. Although the students were informed that their results would be used for research purposes, this test was in fact the actual end-of-unit class test. A marking scheme was developed before the test was administered. After the test was administered I marked the scripts then, in order to avoid discrepancy, a second marker, a colleague, marked the test scripts as well. Our marking resulted in less than 5% discrepancy. Students' errors made in the test were then classified into the three categories of conceptual, procedural and technical as noted in Figure 1.

Based on the above results, several students were selected for an interview. These students were selected for various reasons to include those who did very well or very poorly, those with borderline marks and those who are normally very good in mathematics but did not score high marks in this test. All the interviews were conducted in the school library. Interviews were audio taped for subsequent analysis. Before the commencement of the interview, each student was given time to look through his or her earlier responses. I used a semi-structured interviewing method whereby I asked certain leading questions which tested the students' understanding of the underlying concepts in the test questions and probed further their understanding based on their responses.

Results and Related Discussion

Degree of Understanding of Integration

The first research question, which focused on the degree of understanding of integration concepts among students, was analysed as per the Additional Mathematics syllabus objectives set out by the University of Cambridge Local Examinations Syndicate (UCLES). The overall test results are presented in Table 1.

Table 1
Summary of test results of students

	Total mark	Mean	Standard Deviation	Number of students who scored (in %)			Percentage Pass
				0 - 49	50 - 99	100	
1a	2	1.25	0.82	10	10	20	75%
1bi	1	0.30	0.45	28	0	12	30%
1bii	2	0.45	0.73	28	6	6	30%
1biii	3	1.40	1.36	21	4	15	48%
2	6	3.03	2.37	17	12	11	58%
3	5	1.73	1.65	32	1	7	20%
4	7	3.20	2.44	29	3	8	28%
5i	2	1.98	0.34	0	1	39	100%
5ii	3	2.28	0.96	8	10	22	80%
5iii	3	1.03	0.98	32	2	6	20%
6i	2	1.35	0.70	5	16	19	88%
6ii	2	1.23	0.90	13	5	22	68%
6iii	2	0.73	0.83	21	9	10	48%

There were altogether six questions in the test but only four objectives to be tested. Hence some questions tested more than one objective. Question one of the test was divided into two parts with the second part further divided into three sub parts. The results in the table show for example that for question 1(a), 30 of the 40 students passed the test (10 in the 50% to 99% range, and 20 answered correctly). Note also that question 5(i), testing students on kinematics and solving algebraic equations, was answered correctly by all the students whereas questions three, 1(b)(i) and 1(b)(ii) were the most poorly answered. Analysis on an objective-by-objective basis follows:

Objective 1: Integration of sums of terms in powers of x excluding $\frac{1}{x}$

Students' understanding of integration of sums of terms in powers of x excluding $\frac{1}{x}$ was not tested in one specific question but was incorporated throughout portions of the questions of the entire paper. For example, in question four, students were expected to be able to integrate the function y = x^2 - 6x + 8 in order to find the ratio of area of region A to area of region B. By examining this question in detail, it was found that nearly all the students were able to integrate the two functions mentioned above with the exception of a few who forgot to put in a constant c. Hence it is reasonable to conclude that students had mastered the technique of integrating sums of terms in powers of x excluding $\frac{1}{x}$.

Objective 2: Integration of functions of the form $(ax+b)^n$ and trigonometric functions

Question 1(a) tested the students on integrating functions of the form $(ax+b)^n$. The percentage pass is 75% and half the number of students scored full marks. From the results, it seemed that the students were reasonably capable of integrating this type of function. Question two, like question 1(a), tested the students on integrating functions of the form $(ax+b)^n$. However, the question incorporated some aspects of coordinate geometry. The scores, with a percentage pass of 58%, are not as good as the scores of question 1(a). This score line resulted because question two required the students to apply integration to solve a coordinate geometry problem, which is more complicated when compared with question 1(a). Despite this difficulty, it appears that students are quite proficient in integrating functions of the form $(ax+b)^n$.

Students' understanding of integration of trigonometric functions was tested in questions 1(b)(i), 1(b)(ii) and 1(b)(iii). For question 1(b)(i), the percentage pass is only 30%. By examining students' responses in detail, it was found that many students had mixed up integration with differentiation. Instead of giving the answer as $\frac{1}{2}\sin(2x-1)$, many students gave the answer as $2\sin(2x-1)$. Question 1(b)(ii) required the students to change the original function into another form before performing integration. However, a number of students seemed to forget the trigonometric identity needed for this manipulation. Besides testing integration of trigonometric functions, this question also tested the students on evaluating definite integrals. Similar to question 1(b)(i), many students mixed up integration with differentiation. From the analysis of these three questions, it is reasonable to conclude that the students encountered difficulties in integrating trigonometric functions. Many of them confused integration with differentiation and some could not recall trigonometric identities. They also either forgot the techniques of integrating trigonometric functions or lacked practice in this area.

Objective 3: Evaluation of definite integrals and application of integration to the evaluation of plane areas

Question three tested the students on how to evaluate definite integrals and apply integration to evaluate plane areas. The percentage pass is only 20%. The question, which did not provide any diagrams or sketches, required the students to integrate the curve $y = x(x-4)$ from $x = 0$ to $x = 5$. Because of this, many students failed to realise that the part of the curve $y = x(x-4)$ from $x = 0$ to $x = 4$ is below the x-axis whereas the part from $x = 4$ to $x = 5$ is above the x-axis. Many students had no idea that they need to sketch out the curve to determine how they were going to integrate the function. Many simply integrated the function from $x = 0$ to $x = 5$ directly. They integrated the function mechanically using the given limits.

Question four also focused on evaluating definite integrals and using integration to evaluate plane areas. This time the question provided the diagram and students were supposed to find the ratio of two shaded regions. However, one of the shaded regions lies below the x-axis and students had difficulties finding this area. On the other hand, although the other shaded region lies above the x-axis, students still encountered difficulties as they were not sure what limits to use for integrating or which functions to integrate.

From these two questions, it appears that students are quite competent in evaluating definite integrals but are very weak in applying integration to evaluate plane areas.

Objective 4: Application of differentiation and integration to kinematics problem

Question five involved applying differentiation and integration to solve kinematics problems. Although the present study does not include differentiation, it was included in this question because, in kinematics problems in the Singapore-Cambridge General Certificate in Education examination, these two topics tend to be entwined. Question 5(i) and 5(ii) did not test the students on integration. Question 5(iii) is the question that really dealt with integration. By examining the question in detail, it was found that students had no problem integrating the given function. However, they faltered when they were finding the distance traveled in the first eight seconds as they failed to take into consideration the change in direction of the particle within the first eight seconds. This accounted for the low percentage pass of the question.

Question six, like question five, tested the students on applying differentiation and integration to solve kinematics problems. However, the function involved in this question is a trigonometric function. Question 6(iii) is the only question that tested the students on integration. Again the low percentage pass of this question suggests that the students are not adept with integrating trigonometric functions.

From the results of these two questions, it may be concluded that students are better at integrating polynomial functions as compared to integrating trigonometric functions.

Error analysis

The second research question investigated the kinds of common errors and misconceptions that students encounter when they learn integration. The errors encountered were categorised into conceptual, procedural and technical errors. Table 2 provides a detailed summary of the number of different types of errors committed in each question. Note, for example, that the greatest number of conceptual errors occurred in questions three and four which related to objective three. The other questions contain very few conceptual errors or no conceptual errors at all. The no errors column refers to questions where students did not commit any errors whereas the zero errors column refers to questions that students left blank totally.

Conceptual errors

From the analysis of students' responses to the test questions, it was found that there were altogether three main types of conceptual errors among the 56 conceptual errors made by the students.

Table 2

Summary of the number of different types of errors made in each question (N = 40)

	Conceptual Errors	Procedural Errors	Technical Errors	No Errors	Zero (Blank)
1a	0	17	3	20	0
1bi	0	28	0	12	0
1bii	0	6	28	6	0
1biii	0	18	6	15	1
2	3	12	10	13	2
3	32	0	0	7	1
4	21	1	9	9	0
5i	0	0	1	39	0
5ii	0	0	17	22	1
5iii	0	12	22	6	0
6i	0	0	19	19	2
6ii	0	1	13	22	4
6iii	0	23	2	10	5
Total	56	118	130	200	16

Conceptual Error 1: Integration as area under the curve 1

The first type of conceptual error occurred in question three of the test. Students were supposed to find the area between the curve y = x(x – 4) and the x-axis from x = 0 to x = 5. No diagram was given to aid the students in solving this question. Of the 40 students who took the test, 25 students integrated the curve y = x(x – 4) from x = 0 to x = 5 directly without realizing that the part of the curve from x = 0 to x = 4 is below the x-axis whereas the part from x = 4 to x = 5 is above the x-axis. To illustrate this point, see written sample 1.

<u>Written Sample 1: Student BH</u>

Area between the curve y = x(x – 4) and the x-axis from x = 0 to x = 5 is

$$\int_0^5 x(x-4)dx = \int_0^5 (x^2 - 4x)dx$$

$$= \left[\frac{x^3}{3} - 2x^2\right]_0^5$$

$$= \frac{125}{3} - 50$$

$$= -8\frac{1}{3}\ units^2$$

The same difficulty was also highlighted during an interview with student YF on question 3, reproduced in Vignette 1.

Vignette 1

> YF:. . . it did not cross my mind that I need to sketch out the curve in order to know how to find the required area. I thought I just needed to integrate the function using the given limits.

Besides the type of conceptual error described above, there were five students who recognized that part of the curve lies above and part of it lies below the x-axis but just integrated the curve from x = 0 to x = 4 only. They ignored or were unable to deal with the part from x = 4 to x = 5. An example of this type of misconception was provided by SF's response to this question. See written sample 2.

Written Sample 2: Student SF

Area between the curve y = x(x – 4) and the x-axis from x = 0 to x = 5 is

$$\int_0^4 x(x-4)dx = \int_0^4 (x^2 - 4x)dx$$

$$= \left[\frac{x^3}{3} - 2x^2\right]_0^4$$

$$= \frac{64}{3} - 32$$

$$= -10\frac{2}{3}\,units^2$$

During the interview with SF, she revealed that she thought the part of the curve from x = 4 to x = 5 is not important and does not affect the overall answer. Hence she ignored it totally. From this question, it may be concluded that students experienced difficulties when part of the curve lies above the x-axis and part of it lies below.

Conceptual Error 2: Integration as area under the curve 2

The second type of conceptual error occurred in question four. Students were required to find the ratio of two shaded regions. A diagram was given to help the students understand the question better. However, as one of the shaded regions lies below the x-axis the students had problems finding this area. As a result they integrated using the wrong limits or the wrong functions. On the other hand, although the other shaded region lies above the x-axis, students still encountered difficulties as they again were not sure of what limits to use for integrating or what functions to integrate. Students tried various methods to find the area of the two required shaded regions. An example of this type of error is illustrated by CXY's response in written sample 3.

Written Sample 3: Student CXY

Area of shaded region B $= \int_2^4 (8-2x)dx - \int_2^4 \left(x^2 - 6x + 8\right)dx$

$= \left[8x - x^2\right]_2^4 - \left[\frac{1}{2}x^3 - 3x^2 + 8x\right]_2^4$

= [(32 - 16) - (16 - 4)] - [(32 – 48 + 32) - (4 – 12 + 16)]

= 4 - 8

$= -8\,units^2$

Altogether 21 students committed this type of conceptual error. This once again illustrated students' difficulties when part of the curve lies above the x-axis and part of it lies below.

Conceptual Error 3: Integration as anti-differentiation

The third type of conceptual error arose in question two of the test. The question tested the students' concept of integration as anti-differentiation plus some aspects of coordinate geometry. Students were given the gradient function of a curve and were supposed to integrate this gradient function to get the function of the curve. However, there were three students who did not realize that they needed to integrate the gradient function. Instead, they thought the function of the curve was a straight line and used some other wrong method to find the equation of the curve. To illustrate this point, see written sample 4.

Written Sample 4: Student RAB

Given $\frac{dy}{dx} = \frac{6}{(2x-3)^2}$

When x = 3, $\frac{dy}{dx} = \frac{6}{\left[2(3)-3\right]^2} = \frac{2}{3}$

Since the point (3, 5) lies on the curve, equation of curve is:

$y - 5 = \frac{2}{3}(x - 3)$

$y = \frac{2}{3}x + 3$

When the curve crosses the x-axis, y = 0:

$0 = \frac{2}{3}x + 3$

$x = -4\frac{1}{2}$

Therefore the curve crosses the x-axis at the point ($-4\frac{1}{2}$, 0).

From the above, it may be concluded that some students did not understand integration as anti-differentiation.

Procedural errors

There were altogether 118 procedural errors made by the students. These procedural errors can be classified into two main categories.

Procedural Error 1: Constant c in indefinite integration

The first type of procedural error refers to situations where students integrated indefinite integrals without adding a constant c. Altogether 37 errors were of this nature. An example of this was seen from JBJ's response in written sample 5.

<u>Written Sample 5: Student JBJ</u>

$$\int \cos(2x-1)dx = \frac{1}{2}\sin(2x-1)$$

During the interview with JBJ, she indicated that she had forgotten about the constant c and this error occurred throughout her whole test paper. Besides the error regarding the constant c, there were another eight cases in question 6(iii) where the students did add a constant c but did not go on to find the exact value of c. From the results, it may seem that there were still students who were not aware or might have forgotten that they needed to add a constant c whenever they were doing indefinite integration. As for the eight cases where the students did not find the value of c, they might have misunderstood the question and thought that it was not necessary to find c explicitly.

Procedural Error 2: Confusion over differentiation and integration

The second type of procedural error refers to errors made by students when they confused differentiation with integration. There were 73 errors of this nature. Many students actually used procedures for doing differentiation with integration. There were a number of instances where they multiplied the answer by a certain coefficient when they should have divided and vice versa. Another situation was where they added a negative sign to the answer when they should not have and vice versa. These type of errors occurred mainly in questions which involved integrating trigonometric functions. An example of this type of error was seen from CXY's response in written sample 6.

<u>Written Sample 6: Student CXY</u>

$$\int \cos(2x-1)dx = -2\sin(2x-1) + c$$

Besides her written response, a portion of the interview with CXY is reproduced in Vignette 2.

Vignette 2

I:	Can you explain your answer to this question?
CXY:	Should not have a negative sign.
I:	What else?
CXY:	Then don't need to times the 2.
I:	What should you times then?
CXY:	Should divide by 2.
I:	How come now you know?
CXY:	Because at that time I got confused with differentiation.

It may be concluded from the errors that students were confused with the procedures of performing differentiation and integration, especially with regards to trigonometric functions.

Technical errors

Technical errors refer to errors due to lack of mathematical content knowledge in topics other than integration or errors due to carelessness. Although not specific to integration, technical errors are also of importance to the study because in the GCE 'O' Level Examinations, questions pertaining to integration normally incorporate other topics in the Additional Mathematics syllabus. In all, there were 130 technical errors made by the students, which were classified into five main categories.

Technical Error 1: Lack of knowledge in coordinate geometry

The first technical error is the lack of mathematical content knowledge in coordinate geometry. There were three instances in question two where students thought that the equation of the x-axis was x = 0 instead of y = 0. Besides this error, there were eight instances in question four where students had mistaken the part of the curve y = x^2 - 6x + 8 from x = 0 to x = 2 to be a straight line. This was exemplified by HHL's response in written sample 7.

Written Sample 7: Student HHL

$$\text{Area of shaded region A} = \left(\frac{1}{2}\times 4\times 8\right)-\left(\frac{1}{2}\times 2\times 8\right)$$
$$= 8\ units^2$$

Due to these errors, they were unable to answer the question correctly. These errors had arisen from students' lack of content knowledge in coordinate geometry.

Technical Error 2: Lack of knowledge in kinematics
The second technical error relates to lack of mathematical content knowledge in motion of a particle traveling in a straight line. This type of technical error appeared in question 5(iii). Students were required to find the distance traveled by a particle moving in a straight line in the first eight seconds of its motion. However, there were 21 cases where students ignored the fact that the particle changed direction at t = 5 seconds. As a result, they calculated the wrong distance. Besides the error just described, there were nine instances in question 5(ii) where students did not understand the significance of negative acceleration and were unable to answer the last part. A typical wrong answer given would be: "The particle is moving in the negative direction". These errors arose from their lack of content knowledge in kinematics.

Technical Error 3: Lack of knowledge in algebra
The third technical error focuses on the lack of mathematical content knowledge in algebra. There were two instances in question 1(a) where students made an error in algebra which affected their ability to integrate the function correctly. This was exemplified by SF's response in written sample 8.

<u>Written Sample 8: Student SF</u>

$$\int 2(3+4x)^4 dx = \int (6+8x)^4 dx$$
$$= \frac{(6+8x)^5}{5\times 8} + c$$
$$= \frac{1}{40}(6+8x)^5 + c$$

Besides question 1(a), there were three other separate instances in the other questions where students' errors in algebra affected their ability to get the correct answers.

Technical Error 4: Lack of knowledge in trigonometry
The fourth technical error relates to lack of mathematical content knowledge in trigonometry. There were 36 errors of this nature. Of these, 28 were due to students' lack of knowledge in trigonometric identities. This occurred in question 1(b)(ii) where, in order to be able to integrate, students had to change the trigonometric identity from $\tan^2 2x$ to $\sec^2 2x - 1$. Only a small number of students remembered the required formula needed. The other eight errors were due to students' lack of knowledge in solving trigonometric equations.

Technical Error 5: Carelessness
The fifth technical error category is errors due to carelessness. There were 33 such cases where students had either copied the question wrongly or made some careless

mistakes that resulted in unnecessary loss of marks. For example, YF copied the function as y = x(x + 4) instead of y = x(x – 4) in question three.

Error summary

From the results of the test and the subsequent interviews, a list of the significant types of errors made by students was compiled and is shown in Table 3. Note, for example, there were 53 conceptual errors made by students where they were unable to find areas when the curve crossed the x-axis.

With regards to errors and misconceptions, there were a large number of conceptual errors made by students. These conceptual errors were mainly found in questions three and four which, as noted earlier, related to applying integration to evaluate plane areas. This finding is consistent with Orton's (1983a) observation where he concluded that students in his study had problems with finding areas when the curve crossed an axis or generally in understanding the relationship between a definite integral and areas under the curve.

Table 3
Significant types of errors made by students

Errors	Descriptions	N
Conceptual Errors	Unable to find areas when the curve crossed the x-axis	53
	Failure to realise the need to integrate the gradient function in order to get the function of the curve	3
Procedural Errors	Failure to put a constant c when finding indefinite integrals or failure to evaluate the constant c when necessary	45
	Confusion over differentiation and integration	73
Technical Errors	Coordinate Geometry	12
	Kinematics	32
	Algebra	5
	Trigonometry	36
	Carelessness	33

The students in this study made nearly twice as many procedural errors as conceptual errors. One main reason for this large number of procedural errors was that students mixed up algorithms for performing integration with differentiation. Another major factor was due to the constant c in indefinite integrals which the students failed to include as part of their answers. The high occurrence of procedural errors might also be a result of a lack of practice on the students' part as procedural errors were usually committed while performing algorithms.

Surprisingly, technical errors accounted for the largest number of errors made in the test. This was largely due to the students' lack of mathematical content knowledge in other topics. This came as a surprise because students were expected to have reached a certain level of competency in the other topics. It might imply that students were either generally weak in their mathematics or they had forgotten what they learnt previously. Another potential explanation is that students could perform an operation in isolation, but when embedded in other tasks, they suffered from a form of "cognitive-overload". The high number of technical errors indicated that it is an area of concern that needs to be addressed.

Implications for teaching

From the results of the study, it was found that students generally lacked both conceptual and procedural understanding of integration, and were also quite deficient in their mathematical content knowledge in other topics required for integration. Examining how we teach and how students learn is thus essential. To illustrate this point, two examples of recommendations may be helpful: In the "One Day Conference on Challenges of Calculus Education in Singapore" conducted by the National Institute of Education in Singapore (Ahuja et al., 1998), one recommendation made was to give more emphasis to the understanding of basic concepts than familiarity with techniques only. In other words, teachers, in teaching the topic on integration, could develop concepts first before embarking on techniques in problem solving. Students need to conceptualise first before applying the formulae.

Secondly, Orton (1983a), in his study regarding integration, proposed providing illustrations like diagrams and graphs wherever possible to help students overcome the problem with understanding the relationship between a definite integral and areas under the curve. This point on diagrams was also mentioned in the conference where a recommendation was made regarding using diagrams to illustrate basic concepts. Coincidently, visual thinking was also advocated by Ferrini-Mundy and Lauten (1994) and Eisenberg (1992). They felt that visual thinking in calculus should be promoted to aid in students' understanding of calculus concepts which could be done through making connections between functions and their graphs. Visual illustrations do indeed facilitate students' understanding of calculus concepts and should be used more often by teachers in their lessons. However, it should also be noted that in question four of the test, a diagram was actually provided for the students but they still encountered problems. Thus there may be certain limitations to the benefits of visual illustrations.

The interviews provided deeper insights into students' thinking. This point highlights the importance of communication skills in mathematics. Communication skills in mathematics include reading, writing, speaking, modeling, and so on. These are important skills as they force students to express their ideas clearly and to organize them coherently. Hence teachers could ask students to write down what they think about certain concepts or ideas and then discuss them with their peers. Students could also be asked to set questions as this requires higher order thinking skills. These questions could then be used as a platform for a class discussion where other students can comment on these questions' viability and difficulty level. However, it should be noted that Porter and Masingila's (1995) study on the effects of writing to learn mathematics on the types of conceptual and procedural errors made by students in calculus problems revealed no significant benefits of writing activities. Their study indicated a need for further research into this area.

From the results of the study, it was found that there were a high number of procedural errors. This area cannot be neglected with respect to implications for teaching. Literature reviews tend not to focus on procedural errors. Hence one can only speculate on how to enhance procedural understanding in integration. Since a large number of procedural errors arose from confusion between integration and differentiation processes, one possible solution could be to ask students to compare algorithms or state the difference between algorithms relating to these two processes. Teachers might also stress the importance of the constant c in indefinite integrals to their students.

As for the surprisingly high numbers of technical errors, it raises some questions pertaining to the prerequisite knowledge needed in integration. Which topics should be reviewed before introducing integration? Can the time be found to do so given the time constraints? Solutions may involve conducting remedial lessons after school hours which concentrate on the prerequisite knowledge needed in integration, and preparing revision worksheets on the prerequisite knowledge and incorporating them into normal lessons.

Finally, the notion of limit is a very important concept in integration (Tall & Vinner, 1981; Davis & Vinner, 1986; Bezuidenhout, 2001). There is also a chapter dedicated to limits in many Additional Mathematics textbooks. However, the limit concept tends not to be taught in schools in Singapore because it is not tested in the GCE 'O' Level Examinations. This lack of limit-concept knowledge inevitably affected students' conceptual knowledge of integration thus resulting in misconceptions. Hence teachers could perhaps place more emphasis on the limit concept. By doing so, hopefully their students would be able to gain a better conceptual understanding of integration.

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Appendix
Test on Integration

Q1. (a) Find $\int 2(3+4x)^4\,dx$. [2]

(b) Evaluate

(i) $\int \cos(2x-1)dx$, [1]

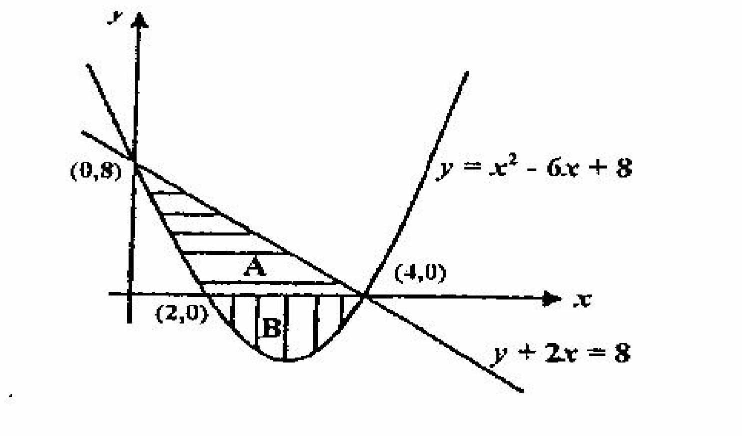
(ii) $\int \tan^2 2x dx$, [2]

(iii) $\int_0^{\frac{\pi}{8}} 4\sec^2 2x dx$. [3]

Q2. A curve is such that $\frac{dy}{dx} = \frac{6}{(2x-3)^2}$. Given that the curve passes through the point (3, 5), find the coordinates of the point where the curve crosses the x-axis. [6]

Q3. Find the area between the curve y = x(x – 4) and the x-axis from x = 0 to x = 5. [5]

Q4. The diagram below shows part of the line y + 2x = 8 and part of the curve y = $x^2 - 6x + 8$. Calculate the ratio of the area of the shaded region A to the shaded region B. [7]



Q5. A particle is moving in a straight line with velocity v = $t^2 - 13t + 40$ m/s where t is the time in seconds after passing a fixed point O.

Calculate

(i) the values of t when the particle is instantaneously at rest, [2]

(ii) the acceleration of the particle when t = 2. What can you conclude from the sign of the value found? [3]

(iii) the distance traveled by the particle in the first 8 seconds? [3]

Q6. A particle moves in a straight line so that, at time t seconds after leaving a fixed point O, its velocity, v m/s, is given by v = 15 sin $\frac{1}{3}$ t.

Find

(i) the time at which the particle first has a speed of 10 m/s, [2]

(ii) the acceleration of the particle when t = 0, [2]

(iii) an expression for the displacement of the particle from O in terms of t. [2]

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